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Generalized BGG Resolutions and Blattner's Formula in Type A

Consider the natural action of $\mathrm{GL}_n(\mathbb{C})$ on p covectors and q vectors; by Howe duality, the space of polynomial functions on this space decomposes multiplicity-free under the joint action of $\mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{gl}_{p+q}(\mathbb{C})$. When $n \geq p+q$ (which is known as the stable range), the $\mathfrak{gl}_{p+q}$-modules are generalized Verma modules (GVMs, introduced by Lepowsky), on which the unipotent radical of the Hermitian real form $\mathrm{U}(p,q)$ of $\mathfrak{gl}_{p+q}$ acts freely. When $n < p+q$, however, the structure of these modules is less transparent. Enright and Willenbring (2004) constructed resolutions for them in terms of GVMs. The goal of this paper is to exhibit a remarkable connection between these resolutions and a seemingly quite different situation, namely the K-type multiplicities in certain discrete series of $\mathrm{SU}(n, p+q)$. More precisely, we establish that the signed multiplicities of the GVMs in the resolution coincide with the values of Blattner's formula for the K-type multiplicities in appropriately chosen discrete series representations of $\mathrm{SU}(n, p+q)$.

Keywords: Howe duality, generalized Verma modules, BGG resolutions, discrete series, Blattner's formula.

MSC: 22E47; 05E10.