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Double Contact Conics in Involution

Three coplanar line segments OA, OB, OC are given and three concentric ellipses C_1, C_2, C_3 are defined, so that every two of the segments are conjugate semi-diameters of one ellipse. In previous studies we proved using Analytic Plane Geometry that the problem of finding an ellipse circumscribed to C_1, C_2, C_3 has at most two solutions. The *primary solution* T_1 is always an ellipse. The *secondary solution* T_2 (if it exists) is an ellipse or a hyperbola. We also constructed T_1 using Synthetic Projective Plane Geometry.
This study investigates the existence and the construction of T_2 with Synthetic Projective Geometry, particularly Theory of Involution. We prove that the common diameters of every couple of C_1, C_2, C_3 correspond through an involution f. Criteria of Synthetic Projective Geometry determine whether f is hyperbolic or elliptic. If f is hyperbolic, exactly two double contact conics T_1, T_2 exist circumscribed to C_1, C_2, C_3. T_1 is always an ellipse. T_2 is an ellipse, a hyperbola or a degenerate parabola. The common diameters of T_1, T_2 define the double lines of f. If f is elliptic, still two double contact conics T_1, T_2 exist. Now T_1 is an ellipse circumscribed and T_2 an ellipse inscribed to C_1, C_2, C_3. Regardless of whether f is hyperbolic or elliptic, we construct T_2 using the already constructed ellipse T_1 and the involution f.

Keywords: Mutually conjugate ellipses, double contact conic, elliptic/hyperbolic involution, double rays, Fregier point.

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