Journal of Convex Analysis 32 (2025) 263–290

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Derivative of Inverse Mapping and Chain Rule Formulas for Gateaux Derivatives, Theorems and Counter-Examples

We investigate what happens if Fréchet differentiability is replaced by Gateaux differentiability in several statements from differential calculus. Our main counter-example shows how the formula for derivative of inverse mapping fails when Gateaux differentiability is assumed both for the mapping and its inverse: We construct an involution $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $f(0,0) = (0,0)$, C^∞-smooth off the origin and Gateaux differentiable at $(0,0)$, whose derivative $f'(0,0)$ is singular, and hence $f'(0,0) \circ f'(0,0)$ is not the identity mapping. This mapping simultaneously provides a strong counter-example for the Chain Rule formula with Gateaux derivatives. We also show that the chain rule formula holds true for the Gateaux derivatives in an important special case of mappings between Banach spaces.

Keywords: Chain rule formula, inverse mapping theorem, derivative, Frechet differentiability, Gateaux differentiability, involution.

MSC: 26B10, 46G05, 49J50, 58C20.