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# Betweenness-Induced Convexity in Hyperspaces of Normed Vector Spaces

Using Minkowski addition of sets, we study linear betweenness in the hyperspace $L(X)$ of linearly convex nonempty subsets of a normed real vector space $X$, as well as in the sub-hyperspace $KL(X)$ of compact elements of $L(X)$. We also study the metric betweenness relation induced by the Hausdorff metric on the latter. While linear betweenness in $L(X)$ behaves reasonably like linear betweenness at the point level, the analogy is not perfect: linear intervals in $X$ are honest line segments; this is no longer the case for $L(X)$, where linear intervals can have exactly two elements. However, when we restrict our focus to $KL(X)$, the Rådström extension theorem allows us to view this hyperspace as a linearly convex cone in a normed vector space $\mathcal{R}(X)$; in particular, all linear intervals are line segments that are contained in the corresponding metric intervals.

We are especially interested in the notions of convexity induced by these two kinds of betweenness relation. While all closed balls and metric intervals in $KL(X)$ are linearly convex, metric convexity has more nuanced behaviour. For example, the metric intervals in $KL(X)$ determined by singletons are all metrically convex if and only if $X$ is strictly convex. When $X$ is one-dimensional, $\mathcal{R}(X)$ is Cartesian 2-space equipped with the *max* norm and $KL(X)$ looks like the half-plane $\{\langle x, y\rangle : x \leq y\}$. In particular, all metric intervals – and no closed balls of positive radius – are metrically convex. When $X$ is multi-dimensional, though, while it is still the case that closed balls are metrically nonconvex, it is now always possible to find a metrically nonconvex metric interval that is determined by a singleton and a line segment.

**Keywords**: Betweenness, betweenness axioms, convexity, vector spaces, Minkowski addition, metric spaces, normed vector spaces, strict convexity, hyperspaces, Hausdorff metric, Radstroem extension.

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